



ACADEMIC  
PRESS

Available online at [www.sciencedirect.com](http://www.sciencedirect.com)

SCIENCE @ DIRECT®

J. Math. Anal. Appl. 275 (2002) 161–164

*Journal of*  
MATHEMATICAL  
ANALYSIS AND  
APPLICATIONS

[www.academicpress.com](http://www.academicpress.com)

# On the state space of $C^*$ -algebras<sup>☆</sup>

Kourosh Nourouzi

*Department of Mathematics, K.N. Toosi University of Technology, P.O. Box 15875-4416, Tehran, Iran*

Received 4 May 2001

Submitted by J. Diestel

---

## Abstract

It is proved that every pure state on a  $C^*$ -algebra  $A$  is an exposed point in the state space of  $A$ .

© 2002 Elsevier Science (USA). All rights reserved.

*Keywords:* Exposed point; State space

---

If  $C$  is a convex set in a complex topological vector space  $X$  with topology  $\tau$ , then an element  $\omega \in C$  is a  $\tau$ -*exposed point* of  $C$  if there is a  $\tau$ -continuous linear map  $f : X \rightarrow \mathbb{C}$  such that  $\operatorname{Re} f(\gamma) < \operatorname{Re} f(\omega)$ , for every  $\gamma \in C \setminus \{\omega\}$ . Thus,  $\omega$  is an exposed point of  $C$  (relative to the given topology  $\tau$ ) if there is a supporting hyperplane for  $C$  that passes through  $\omega$  but does not contain any other points of  $C$ . In the case where  $X$  is a Banach space and  $\tau$  is the metric topology that is induced by the norm on  $X$ , then  $\tau$ -exposed points of  $C$  are simply called *exposed* points. A  $\tau$ -exposed point of a convex set is necessarily an extreme point of that set, but there are convex sets with nonexposed extreme points.

Some convex sets of interest in operator theory are the unit balls in a  $C^*$ -algebra, its predual, and its dual; the extreme and exposed points of these sets have been studied by Akemann and Russo in [1]. Similarly, the exposed points of the unit ball of bounded operators on Hilbert spaces have been characterised by Grzaslewicz [2].

---

<sup>☆</sup> This work is supported in part by the Ministry of Higher Education of Iran.

In the present note it is proved that every pure state is exposed in the state space of a  $C^*$ -algebra.

### States on $C^*$ -algebras

Let  $A$  denote a  $C^*$ -algebra and let  $A^+$  be the positive cone of  $A$ . The *state space* of  $A$  is the weak\*-compact convex set of all norm-continuous linear maps  $\varphi : A \rightarrow \mathbb{C}$  such that  $\varphi(A^+) \subseteq \mathbb{R}_0^+$  and  $\|\varphi\| = 1$ . By the Krein–Milman theorem,  $S(A)$  has extreme points; these are called *pure* states.

In [1], Akemann and Russo prove the following interesting result.

**Theorem A** (Akemann and Russo). *The following statements are equivalent if  $A$  is a separable unital  $C^*$ -algebra.*

- (a)  $\varphi$  is an extreme point of  $S(A)$ .
- (b)  $\varphi$  is an exposed point of  $S(A)$ .
- (c)  $\varphi$  is a weak\*-exposed point of  $S(A)$ .

*Moreover, there are nonseparable abelian  $C^*$ -algebras that possess at least one pure state that is not weak\*-exposed.*

The purpose is to draw attention to the fact that statements (a) and (b) are equivalent for every  $C^*$ -algebra.

The enveloping von Neumann algebra  $A''$  of  $A$  contains (an  $*$ -isomorphic copy of)  $A$  as a  $C^*$ -subalgebra and every state  $\varphi$  on  $A$  has a unique extension to a normal state  $\tilde{\varphi}$  on the von Neumann algebra  $A''$ . The dual space  $A^*$  is spanned by states on  $A$  and  $A^*$  is isomorphic to the Banach space  $(A'')_*$  of normal states on  $A''$ . Thus,  $A^{**}$  can be identified with the von Neumann algebra  $A''$ . These identifications are explained in [3]. In particular, the following result from [3] will be useful.

**Lemma B** [3, 3.10.7, 3.13.6]. *Let  $A$  be a  $C^*$ -algebra and  $A''$  its enveloping von Neumann algebra. If  $\varphi$  is a pure state on  $A$  and if  $L_\varphi$  denotes the set  $\{x \in A : \varphi(x^*x) = 0\}$ , then*

- (1)  $\ker \varphi = L_\varphi + L_\varphi^*$ , where  $L_\varphi^*$  denotes the set  $\{x \in A : x^* \in L_\varphi\}$ , and
- (2) *there is a projection  $p \in A''$  such that  $1 - p$  is a minimal projection in  $A''$ ,  $\tilde{\varphi}(p) = 0$ , and  $L_\varphi = \{x \in A : x = xp\}$ .*

**Theorem 1.** *A state on a  $C^*$ -algebra is pure if and only if it is exposed in the state space.*

**Proof.** We need only prove that pure states are exposed.

Let  $\varphi$  be a pure state on a  $C^*$ -algebra  $A$ . By Lemma B, there is a projection  $p \in A''$  such that  $\tilde{\varphi}(p) = 0$  and  $L_\varphi = \{x \in A: x = xp\}$ . Let  $\sim$  be the isometric isomorphism from  $A^*$  onto  $(A'')_*$  and define a linear mapping  $f: A^* \rightarrow \mathbb{C}$  by  $f(\theta) = \tilde{\theta}(1 - p)$ , for every  $\theta \in A^*$ . Via the isometric isomorphism between the Banach spaces  $A^{**}$  and  $A''$ , this linear map  $f$  is norm-continuous; hence,  $f \in A^{**}$ .

Observe that  $f(\varphi) = 1$ ; if  $\psi \in S(A)$ , then

$$f(\psi) = \tilde{\psi}(1 - p) \leq \|\tilde{\psi}\| \|1 - p\| = 1.$$

Thus,  $f$  is a support functional for  $S(A)$  at  $\varphi$ . Let  $E_\varphi = \{\psi \in S(A): f(\psi) = 1\}$ . Because  $E_\varphi$  is the intersection of the norm-closed set  $f^{-1}(\{1\})$  and the weak\*-closed set  $S(A)$ ,  $E_\varphi$  is weak\*-closed and, hence, weak\*-compact. Furthermore,  $E_\varphi$  is plainly convex, and so  $E_\varphi$  is determined by its extreme points. It is sufficient to show that  $\psi = \varphi$ , for any extreme point  $\psi$  of  $E_\varphi$ .

Let  $\psi$  be an extreme point of  $E_\varphi$ . If  $\psi = \frac{1}{2}(\varphi_1 + \varphi_2)$ , for some  $\varphi_1, \varphi_2 \in S(A)$ , then

$$f(\psi) = \frac{1}{2}(f(\varphi_1) + f(\varphi_2)),$$

which implies that  $\varphi_1, \varphi_2 \in E_\varphi$ . Hence,  $\psi = \varphi_1 = \varphi_2$ , which is to say that  $\psi$  is an extreme point of  $S(A)$ .

Because  $\psi$  is a pure state,  $\ker \psi = L_\psi + L_\psi^*$ . If  $x \in L_\varphi$ , then  $x = xp$  and so  $0 \leq \psi(x^*x) = \tilde{\psi}(p(x^*x)p) \leq \|x\|^2 \tilde{\psi}(p) = 0$ . Thus,  $L_\varphi \subseteq L_\psi$ , implying that  $\ker \varphi \subseteq \ker \psi$  (Lemma B). The kernel of any nonzero linear functional on  $A$  has co-dimension 1, and so  $\varphi$  and  $\psi$  are linearly dependent. But  $\varphi$  and  $\psi$  are positive preserving; hence,  $\varphi = \psi$ .  $\square$

Weak\*-exposed states are somewhat harder to identify because in the case of nonseparable algebras there can be states that are exposed but not weak\*-exposed. The following result is of interest in that it provides easy examples of weak\*-exposed points for some separable and nonseparable algebras. We shall denote  $B(H)$  and  $K(H)$ , the  $C^*$ -algebra of all bounded operators and compact operators on a Hilbert space  $H$ , respectively.

**Theorem 2.** *If  $A \subseteq B(H)$  is a  $C^*$ -subalgebra with  $K(H) \subseteq A$ , then every vector state  $x \mapsto \langle x\xi, \xi \rangle$  on  $A$ , for some fixed unit vector  $\xi \in H$ , is a weak\*-exposed point of the state space of  $A$ .*

**Proof.** Let  $\xi \in H$  be a unit vector and consider the vector state  $\varphi(a) = \langle a\xi, \xi \rangle$ , for  $a \in A$ . Let  $p = \xi \otimes \xi$ , the projection onto  $\text{Span}\{\xi\}$ , and define a linear map  $f: A^* \rightarrow \mathbb{C}$  by  $f(\psi) = \psi(p)$ , for  $\psi \in A^*$ . Then  $f$  is weak\*-continuous and, if  $\psi$  is a state on  $A$ , then  $0 \leq f(\psi) = \psi(p) \leq \|\psi\| \|p\| = 1$ . In addition,  $f(\varphi) = \varphi(p) = \langle p\xi, \xi \rangle = 1$ . Thus,  $f$  is a weak\*-continuous support functional for  $S(A)$  at  $\varphi$ .

As in the proof of Theorem 1, if  $E_\varphi = \{\psi \in S(A) : f(\psi) = 1\}$ , then  $E_\varphi$  is convex and weak\*-compact, and every extreme point  $\psi$  of  $E_\varphi$  is also a pure state of  $A$ .

Let  $\psi(\cdot) = \langle \pi(\cdot)\xi_\psi, \xi_\psi \rangle$  denote the Gelfand–Naimark–Segal decomposition of  $\psi$ . Note that  $\pi$  is an irreducible \*-representation of  $A$  on a Hilbert space  $H_\pi$  (because  $\psi$  is a pure state). Thus, either  $K(H) \subseteq \ker \pi$  or  $\pi$  is unitarily equivalent to the identity representation  $A \rightarrow B(H)$ . Because  $p \in K(H)$  and  $\psi(p) = 1$ , it cannot be that  $K(H) \subseteq \ker \pi$ ; thus, there is a unitary operator  $u : H \rightarrow H_\pi$  such that  $\pi(a)u = ua$ , for all  $a \in A$ . Hence, there is a unit vector  $\gamma \in H$  such that  $u\gamma = \xi_\psi$ , and so, for every  $a \in A$ ,

$$\psi(a) = \langle \pi(a)u\gamma, u\gamma \rangle = \langle ua\gamma, u\gamma \rangle = \langle a\gamma, \gamma \rangle.$$

Thus,  $1 = \psi(p) = \langle p\gamma, \gamma \rangle \leq 1$ , which implies that  $\gamma \in \text{Span}\{\xi\}$  and, finally, that  $\psi = \varphi$ .  $\square$

From the theorem above and by the fact that all pure states on  $K(H)$  are vector states, one the following result is immediate.

**Corollary.** *Every pure state of  $K(H)$  is weak\*-exposed.*

## Acknowledgment

The author would like to thank professor Douglas R. Farenick for much more his helpful comments on this note.

## References

- [1] C.A. Akemann, B. Russo, Geometry of the unit sphere in a  $C^*$ -algebra and its dual, Pacific J. Math. 32 (1970) 575–585.
- [2] R. Grzaslewicz, Exposed points in the unit ball of  $\mathcal{L}(H)$ , Math. Z. 193 (1986) 595–596.
- [3] G.K. Pedersen,  $C^*$ -Algebras and their Automorphism Groups, Academic Press, New York, 1979.